ON THE HOMOGENIZATION OF CONTROL SYSTEM WITH NON-REGULAR CONSTRAINTS

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This paper is devoted to the homogenization problem of a control objects all components of mathematical description of which may depend on some small parameter ε. It is assumed that the control object is discribed by a linear elliptic equation subject to control constraints. As it is well known there is a huge amount of literature on various aspects and methods in homogenization of partial differential equations and operator equations in Banach spaces (see, e.g., [1-7]). While only few papers deal with the homogenization of control systems. That's why the aim of this paper is to study the passing to the limit in such objects as $\varepsilon \to 0$. We will try to find out what happens to the control object as $\varepsilon \to 0$, does there exist a limit, and, if so, can it be determined? In order to do it we note that each of the control system can be characterized by its own set of admissible pairs "control-state". Therefore we will study the homogenization problem as identification of the (Painleve–Kuratowski) topological limit [8] of the collection of sets of admissible pairs .

Let Ω be a bounded open set of R^n with Lipschitz boundary. We define the control object as follows

$$-\operatorname{div}(A_\varepsilon \nabla y) = b_\varepsilon u + f_\varepsilon \quad \text{in } \Omega, \tag{1}$$

$$y = 0 \quad \text{on } \partial\Omega, \quad u \in U_\varepsilon. \tag{2}$$

Let us denote by $w_{H_0^1}$ the weak topology of $H_0^1(\Omega)$, w_{L^2} the weak topology of $L^2(\Omega)$, $s_{H^{-1}}$ the strong topology of $H^{-1}(\Omega)$, and let us begin with the following assumptions:

(1) $\{U_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is a family of weakly closed convex subsets of $L^2(\Omega)$ such that there exists a non-empty topological limit $(w_{L^2})-\mathrm{Lm}\,U_\varepsilon$ in the Kuratowski's sense;

(2) the sequence $\{f_\varepsilon \in H^{-1}(\Omega)\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to the weak topology of $H^{-1}(\Omega)$;

(3) the sequence $\{b_\varepsilon \in L^\infty(\Omega)\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to the strong topology of $L^\infty(\Omega)$;

(4) $A_\varepsilon \in [L^\infty(\Omega)]^{n^2}$ for every $\varepsilon \in (0,\varepsilon_0]$, and there are two positive constants $0 < \beta_0 \le \beta_1$ satisfying $\beta_0 |\xi|^2 \le (\xi, A_\varepsilon \xi)_{R^n} \le \beta_1 |\xi|^2$, a.e. in Ω for any $\xi \in R^n$ and $\varepsilon \in (0,\varepsilon_0]$;

(**5**) boundary problem (1)–(2) is the uniformly regular, i.e. for every ε

$$\Xi_\varepsilon = \left\{ (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{c} -\mathrm{div}\left(A_\varepsilon \nabla y\right) = b_\varepsilon u + f_\varepsilon, \text{ in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \\ u \in U_\varepsilon, \end{array} \right\} \neq \emptyset.$$

It is well known that under above conditions there exists unique solution $y_\varepsilon \in H_0^1(\Omega)$ of original system (1) for every admissible control $u \in U_\varepsilon \subset L^2(\Omega)$. Our aim is to establish the sufficient conditions under which the topological limit of the sets $\{\Xi_\varepsilon\}$ in the $\mu = w_{L^2} \times w_{H_0^1}$ -topology for the product space $L^2(\Omega) \times H_0^1(\Omega)$ can be recovered. In order to do it we will use the following result.

LEMMA 1. *A set E is the tological limit of the sequence*

$$\{E_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]} \subset X$$

in some topology τ if and only if the following conditions are satisfied:

(i) *for every $x \in E$ there exist an index set $H \in \mathbf{H}$ and a sequence $\{x_\varepsilon\}_{\varepsilon \in H}$ converging to x in X such that $x_\varepsilon \in X_\varepsilon$ for every $\varepsilon \in H$;*
(ii) *if H is any index set of $\mathbf{H}^\sharp$, $\{x_\varepsilon\}_{\varepsilon \in H}$ is a sequence converging to x in X such that $x_\varepsilon \in E_\varepsilon$ for every $\varepsilon \in H$, then $x \in E$.*

Here $\mathbf{H}$ is a filter on $(0, \varepsilon_0]$, and $\mathbf{H}^\sharp$ is the grill associated with the filter $\mathbf{H}$, i.e., the family of subsets of $(0, \varepsilon_0]$ that meet all sets H in $\mathbf{H}$.

Let us consider the sequences of operators $\{\mathbf{A}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ and $\{\mathbf{B}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ such that:

(a) $\langle \mathbf{B}_\varepsilon u, \varphi \rangle_{H_0^1(\Omega)} = \int_\Omega b_\varepsilon u \varphi \, dx$, $\forall \varphi \in H_0^1(\Omega)$, i.e. $\mathbf{B}_\varepsilon$ are linear continuous operators from $L^2(\Omega)$ to $H^{-1}(\Omega)$ for every $\varepsilon \in (0, \varepsilon_0]$;
(b) $\left\langle \mathbf{A}_\varepsilon y, \varphi \right\rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla \varphi, A_\varepsilon \nabla y)_{R^n} \, dx$, $\forall y, \varphi \in H_0^1(\Omega)$.

Then original control system (1)–(2) can be rewritten in the form

$$\mathbf{A}_\varepsilon y = \mathbf{B}_\varepsilon u + f_\varepsilon \quad \text{in} \quad D'(\Omega), \quad u \in U_\varepsilon. \tag{3}$$

DEFINITION 1. *We say that a collection of control constraints $\{U_\varepsilon\}$ is the non-regular if $\left(s_{H^{-1}}\right) - \mathrm{Ls}\, Q_\varepsilon = \emptyset$, where by Q_ε denote the images of the sets U_ε in $H^{-1}(\Omega)$ by the maps $F_\varepsilon : L^2(\Omega) \to H^{-1}(\Omega)$. Here $F_\varepsilon u = \mathbf{B}_\varepsilon u + f_\varepsilon$.*

By $\Lambda_\varepsilon \subset L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega)$ we denote the set of all admissible triplet for the problem (3), i.e.

$$\Lambda_\varepsilon = \left\{ (u,g,y) \in L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{c} (g,y) \in \mathrm{gr}\,(\mathbf{A}_\varepsilon)\,|_{Q_\varepsilon \times H_0^1}, \\ g = \mathbf{B}_\varepsilon u + f_\varepsilon, \\ u \in U_\varepsilon, \end{array} \right\} \tag{4}$$

where the graph restriction $\mathrm{gr}\,(\mathbf{A}_\varepsilon)\,|_{Q_\varepsilon \times H_0^1}$ of the operator $\mathbf{A}_\varepsilon$ is defined as the set

$$\mathrm{gr}\,(\mathbf{A}_\varepsilon)\,|_{Q_\varepsilon \times H_0^1} = \mathrm{gr}\,(\mathbf{A}_\varepsilon) \cap \left[Q_\varepsilon \times H_0^1(\Omega)\right],$$

$$\mathrm{gr}\,(\mathbf{A}_\varepsilon) = \left\{ (g,y) \in H^{-1}(\Omega) \times H_0^1(\Omega) \;\middle|\; g = \mathbf{A}_\varepsilon y \right\}.$$

It is easy to prove the following result.

LEMMA 2. *For every $\varepsilon \in (0, \varepsilon_0]$ there is a one–to–one correspondence between the sets Ξ_ε and Λ_ε.*

Now it is easy to see that the problem of topological convergence of the sets of admissible pairs $\{\Xi_\varepsilon\}$ in the μ-topology can be reduced to the identification of topological limit in $\tau = s_{H^{-1}} \times w_{H_0^1}$-topology of the graph restriction sequence $\left\{\mathrm{gr}\,(\mathbf{A}_\varepsilon)\,|_{Q_\varepsilon \times H_0^1}\right\}_{\varepsilon \in (0,\varepsilon_0]}$. However, under our initial assumption (with respect to the non-regular constraints) it is not possible to recover the topological limit of this sequence in the τ-topology, because by virtue of the properties in the Kuratowski's sense, we have the following inclusion

$$\tau\text{–}\mathrm{Ls}\,\mathrm{gr}\,(\mathbf{A}_\varepsilon)\,|_{Q_\varepsilon \times H_0^1} \subseteq \tau\text{–}\mathrm{Ls}\,\mathrm{gr}\,(\mathbf{A}_\varepsilon) \cap \left[(s_{H^{-1}})\,\text{–}\mathrm{Ls}\,Q_\varepsilon \times H_0^1(\Omega)\right] = \emptyset.$$

Consequently, we should choose more weaker topology on $H^{-1}(\Omega) \times H_0^1(\Omega)$ than the τ-topology. With this aim we will consider this problem with respect to τ^*-topology, which is defined as the product of the weak topology for $H^{-1}(\Omega)$ and the weak topology for $H_0^1(\Omega)$. To this we introduce the following hypotheses:

(A1) there exist subsets $L^\varepsilon \subset H^{-1}(\Omega)$ such that $Q_\varepsilon \subseteq L^\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$;

(A2) for every $\varepsilon \in (0, \varepsilon_0]$ there is a real reflexive separable Banach space Y_ε with norm $\|\cdot\|_\varepsilon$ and a continuous linear mapping P_ε of Y_ε into $H_0^1(\Omega)$ such that:

$$\sup_{\varepsilon \in (0,\varepsilon_0]} \|P_\varepsilon\| = c_0 < \infty;$$

(A3) for every $\varepsilon \in (0, \varepsilon_0]$ there exists a linear mapping R_ε^+ of Y_ε^* into $L^\varepsilon \subset H^{-1}(\Omega)$ such that if $g \in Y_\varepsilon^*$, then $P_\varepsilon^*(R_\varepsilon^+ g) = g$ for every $\varepsilon \in (0, \varepsilon_0]$;

(A4) for every strongly converging sequence $\{q_\varepsilon\}$ in $H^{-1}(\Omega)$ we have $\{R_\varepsilon^+ P_\varepsilon^* q_\varepsilon\}$ is bounded.

Now we introduce the following concepts.

DEFINITION 2. *We say that the collection of real reflexive separable Banach spaces $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is coordinated with the control object* (3) *if hypotheses (A1)–(A4) hold true and there is a sequence of convex closed subsets $\left\{\widehat{Q}_\varepsilon \subseteq H^{-1}(\Omega)\right\}_{\varepsilon \in (0,\varepsilon_0]}$ such that $R_\varepsilon^+ P_\varepsilon^* : \widehat{Q}_\varepsilon \to Q_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$, and $s_{H^{-1}}\text{–}\mathrm{Li}\,\widehat{Q}_\varepsilon \neq \emptyset$, whereas $s_{H^{-1}}\text{–}\mathrm{Ls}\,Q_\varepsilon = \emptyset$.*

DEFINITION 3. *For control object* (3) *with a coordinated collection of spaces $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ the sets*

$$\mathrm{Gr}\,(\mathbf{A}_\varepsilon) = \left\{(f, y) \in H^{-1}(\Omega) \times H_0^1(\Omega) \;\middle|\; \mathbf{A}_\varepsilon y = R_\varepsilon^+ P_\varepsilon^* f\right\}$$

are called the prototypes of the operator graphs $\mathrm{gr}\,(\mathbf{A}_\varepsilon)$.

DEFINITION 4. *Suppose $\mathbf{A}_* \in L\left(H_0^1(\Omega); H^{-1}(\Omega)\right)$ is a coercive operator. We say that the sequence of operators $\left\{\mathbf{A}_\varepsilon \in L\left(H_0^1(\Omega); H^{-1}(\Omega)\right)\right\}_{\varepsilon \in (0,\varepsilon_0]}$ G^*-converges to the operator $\mathbf{A}_*$ (in symbols, $\mathbf{A}_\varepsilon \xrightarrow{G^*} \mathbf{A}_*$), if*

$$\tau\text{–}\mathrm{Lm}\,\mathrm{Gr}\,(\mathbf{A}_\varepsilon) = \mathrm{gr}\,(\mathbf{A}_*),$$

where $\tau = s_{H^{-1}} \times w_{H_0^1}$.

We note that definition of the G^*-limit of the operators $\{\mathbf{A}_\varepsilon\}$ is defined in the terms of the product of the strong topology for $H^{-1}(\Omega)$ and the weak topology for $H_0^1(\Omega)$. Moreover, if we put $Y_\varepsilon = H_0^1(\Omega)$, $P_\varepsilon y = y$, $R_\varepsilon^+ g = g$ for every $y \in H_0^1(\Omega)$, $g \in H^{-1}(\Omega)$, and $\varepsilon \in (0, \varepsilon_0]$, then $\widehat{Q}_\varepsilon = Q_\varepsilon$ and each of the graph prototypes $\operatorname{Gr}(\mathbf{A}_\varepsilon)$ coinsides with the corresponding graph $\operatorname{gr}(\mathbf{A}_\varepsilon)$. Therefore Definition 4 reduces to the well known definition of G-convergence. Now we give the following important results.

PROPOSITION 1. *Suppose that for the original control object there is a coordinated collection of Banach spaces* $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$. *Let* $\mathbf{A}_* \in L\left(H_0^1(\Omega); H^{-1}(\Omega)\right)$ *be a coercive operator,* $\left\{\mathbf{A}_\varepsilon \in L\left(H_0^1(\Omega); H^{-1}(\Omega)\right)\right\}_{\varepsilon \in (0,\varepsilon_0]}$ *be a* G^**-compact set of uniformly bounded and uniformly coercive operators. Then the sequence* $\{\mathbf{A}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ G^**-converges to* $\mathbf{A}_*$ *if and only if*

$$\mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* f \longrightarrow \mathbf{A}_*^{-1} f \ \textit{weakly in} \ H_0^1(\Omega)$$

for any $f \in H^{-1}(\Omega)$.

Proof. Assume that $\mathbf{A}_\varepsilon \xrightarrow{G^*} \mathbf{A}_*$. Then, by Definition of G^*-convergence, we have

$$\mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* f \longrightarrow \mathbf{A}_*^{-1} f \ \text{weakly in} \ H_0^1(\Omega),$$

and the "only if" part of the statement is proved.

Let us prove the "if" part. Suppose that $\mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* f \longrightarrow \mathbf{A}_*^{-1} f$ weakly in $H_0^1(\Omega)$ for any $f \in H^{-1}(\Omega)$. By G^*-compactness of the set $\{\mathbf{A}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$, there exists an index set $H \in \mathbf{H}^\sharp$ and a subsequence $\{\mathbf{A}_\varepsilon\}_{\varepsilon \in H}$ such that $\mathbf{A}_{\varepsilon \in H} \xrightarrow{G^*} \widehat{\mathbf{A}}_*$, where $\widehat{\mathbf{A}}_*$ is a linear bounded coercive operator from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Consequently for $\widehat{\mathbf{A}}_*$ there exists an invertible bounded operator $\widehat{\mathbf{A}}_*^{-1}$. The definition of G^*-convergence implies that $\widehat{\mathbf{A}}_*^{-1} f = \mathbf{A}_*^{-1} f$ for any $f \in H^{-1}(\Omega)$. Therefore $\widehat{\mathbf{A}}_*^{-1} = \mathbf{A}_*^{-1}$, and $\widehat{\mathbf{A}}_* = \mathbf{A}_*$. Thus $\mathbf{A}_\varepsilon \xrightarrow{G^*} \mathbf{A}_*$.

THEOREM 1. *Suppose that the following conditions hold true:*

(i) $\left\{\mathbf{A}_\varepsilon \in L\left(H_0^1(\Omega), H^{-1}(\Omega)\right)\right\}_{\varepsilon \in (0,\varepsilon_0]}$ *is a sequence of uniformly coercive and uniformly bounded operators;*

(ii) *the collection of Banach spaces* $\{Y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *is coordinated with the original control object* (3) *in the sense of Definition 2.*

Then there exist an index set $H \in \mathbf{H}^\sharp$, *a subsequence* $\{\mathbf{A}_\varepsilon\}_{\varepsilon \in H}$, *and a coercive linear operator* $\mathbf{A}_*$ *of* $H_0^1(\Omega)$ *into* $H^{-1}(\Omega)$ *such that* $\mathbf{A}_\varepsilon \xrightarrow{G^*} \mathbf{A}_*$, *i.e.*

$$\tau\text{–}\operatorname{Lm}\operatorname{Gr}(\mathbf{A}_\varepsilon) = \operatorname{gr}(\mathbf{A}_*).$$

Proof. Since the space $H_0^1(\Omega)$ is separable and reflexive, there exists a metric d such that for any sequence $\{y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ the following conditions are equivalent:

(1) $y_\varepsilon \to y$ weakly in $H_0^1(\Omega)$;

(2) $\{y_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ is bounded and $d(y_\varepsilon, y) \to 0$.

We denote by σ the topology associated to the metric d on $H_0^1(\Omega)$. This topology has a countable base. Since the topology $s_{H^{-1}} \times \sigma$ has a countable base, by Kuratowski compactness theorem, there exists a subsequence $\{\mathrm{Gr}\,(\mathbf{A}_\varepsilon)\}_{\varepsilon \in H}$, where $H \in \mathbf{H}^\sharp$, such that the one converges to a set $C \subset H^{-1}(\Omega) \times H_0^1(\Omega)$ in the $s_{H^{-1}} \times \sigma$-topology.

Now we prove that $C = \tau\text{–}\mathrm{Lm}\,\mathrm{Gr}\,(\mathbf{A}_\varepsilon)$. With this aim it is enough to show that

$$\tau\text{–}\mathrm{Ls}\,\mathrm{Gr}\,(\mathbf{A}_\varepsilon) \subseteq C, \tag{5}$$

$$C \subseteq \tau\text{–}\mathrm{Li}\,\mathrm{Gr}\,(\mathbf{A}_\varepsilon). \tag{6}$$

Firstly, let us verify (5). Suppose $(f, y) \in \tau\text{–}\mathrm{Ls}\,\mathrm{Gr}\,(\mathbf{A}_\varepsilon)$. Then there exist an index set $H \in \mathbf{H}^\sharp$ and a sequence $\left\{\left(\widehat{f}_\varepsilon, y_\varepsilon\right)\right\}_{\varepsilon \in H}$ converging to (f, y) in the topology τ such that $\left(\widehat{f}_\varepsilon, y_\varepsilon\right) \in \mathrm{Gr}\,(\mathbf{A}_\varepsilon)$ for every $\varepsilon \in H$. Since (1) implies (2), we see that $\left(\widehat{f}_\varepsilon, y_\varepsilon\right)$ converges to (f, y) with respect to the topology $s_{H^{-1}} \times \sigma$. Hence, $(f, y) \in C$.

Now we prove (6). Let $(f, y) \in C$. Then there exists a sequence $\left\{\left(\widehat{f}_\varepsilon, y_\varepsilon\right)\right\}$ converging to (f, y) in the topology $s_{H^{-1}} \times \sigma$ such that $\left(\widehat{f}_\varepsilon, y_\varepsilon\right) \in \mathrm{Gr}\,(\mathbf{A}_\varepsilon)$ for all ε small enough. Since $\left\{\widehat{f}_\varepsilon\right\}$ is bounded in $H^{-1}(\Omega)$ we deduce that the sequence $y_\varepsilon = \mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon$ is bounded in $H_0^1(\Omega)$ as well (by Definition 2). Then the equivalence between conditions (1) and (2) yields weak convergence of $\{y_\varepsilon\}$ to y. Hence, $\left\{\left(\widehat{f}_\varepsilon, y_\varepsilon\right)\right\}_{\varepsilon \in (0, \varepsilon_0]}$ converges to (f, y) in the τ-topology, which implies (6).

Finaly, we prove that there exists an invertible linear bounded operator $\mathbf{A}_*: H_0^1(\Omega) \to H^{-1}(\Omega)$ such that $C = \mathrm{gr}\,(\mathbf{A}_*)$. Using Proposition 1, we see that there exists a linear operator $C_*: H^{-1}(\Omega) \to H_0^1(\Omega)$ such that

$$\forall\, f \in H^{-1}(\Omega) \quad y_\varepsilon = \mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* f \longrightarrow C_* f \text{ weakly in } H_0^1(\Omega).$$

Then by analogy with [9] (see Proposition 1.7) it can be proved that there is a constant $\alpha > 0$ such that the inequalities

$$\|f - g\|^2_{H^{-1}} \le \alpha \|C_* f - C_* g\|^2_{H_0^1}, \tag{7}$$

$$\langle f - g, C_* f - C_* g \rangle \ge \alpha^{-1} \|C_* f - C_* g\|^2_{H_0^1}. \tag{8}$$

hold for every $f, g \in H^{-1}(\Omega)$.

Therefore from (7)–(8) we deduce that for any $f \in H^{-1}(\Omega)$

$$\|f\|^2_{H^{-1}} \le \alpha \|C_* f\|^2_{H_0^1}, \quad \langle f, C_* f \rangle \ge \alpha^{-1} \|C_* f\|^2_{H_0^1}. \tag{9}$$

Consequently the operator C_* is invertible, i.e. we may set $\mathbf{A}_* = C_*^{-1}$. Moreover, we obtain for the operator $\mathbf{A}_*$ the properties of boundedness and coerciveness taking arbitrary $y \in H_0^1(\Omega)$ and substituting $f = \mathbf{A}_* y$ into (9). The theorem is proved.

THEOREM 2. *Suppose that the following conditions hold true:*

(i) $\{\mathbf{A}_\varepsilon \in L(H_0^1(\Omega), H^{-1}(\Omega))\}_{\varepsilon\in(0,\varepsilon_0]}$ *is a sequence of uniformly coercive and uniformly bounded operators;*

(ii) *for the original control object* (3) *there exists a coordinated collection of Banach spaces* $\{Y_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *;*

(iii) *there are an index set* $H \in \mathbf{H}$ *and a* τ*-converging sequence* $\left\{\left(\widehat{f}_\varepsilon, y_\varepsilon\right) \in \widehat{Q}_\varepsilon \times H_0^1(\Omega)\right\}_{\varepsilon\in H}$ *such that*

$$\mathbf{A}_\varepsilon y_\varepsilon = R_\varepsilon^+ P_\varepsilon^* \widehat{f}_\varepsilon, \quad \text{for every } \varepsilon \in H.$$

Then there exist a set $H \in \mathbf{H}^\sharp$*, and a coercive bounded linear operator* $\mathbf{A}_* \in L(H_0^1(\Omega), H^{-1}(\Omega))$ *such that* $\mathbf{A}_\varepsilon \xrightarrow{G^*} \mathbf{A}_*$ *and*

$$\tau\text{--Lm}\left[\mathrm{Gr}(\mathbf{A}_\varepsilon)\Big|_{\widehat{Q}_\varepsilon\times H_0^1(\Omega)}\right] = \mathrm{gr}(\mathbf{A}_*)\Big|_{(s_{H^{-1}})-\mathrm{Lm}[\widehat{Q}_\varepsilon]\times H_0^1(\Omega)}. \tag{10}$$

To prove this theorem we first make use the following result (see [10]).

LEMMA 3. *Let* X, Y *be Banach spaces,* η *be the product topology for* $X\times Y$. *Let* $\{W_\varepsilon\}$ *and* $\{R_\varepsilon\}$ *be some sequences of* η*-closed convex subsets of* $X\times Y$ *for which the following conditions hold:*

(a) $\Pi r_Y W_\varepsilon = Y$ *for every* $\varepsilon\in(0,\varepsilon_0]$, *where by* $\Pi r_Y : X\times Y \to Y$ *denote the projection operator;*

(b) *the sets* R_ε *have representation* $R_\varepsilon = X\times C_\varepsilon$ *for every* $\varepsilon\in(0,\varepsilon_0]$;

(c) *there exist topological limits* $\eta\text{--Lm}\, W_\varepsilon$ *and* $\eta\text{--Lm}\, R_\varepsilon$;

(d) $\eta\text{--Li}\,(W_\varepsilon\cap R_\varepsilon)\neq\emptyset$.

Then for the sequence of subsets $\{W_\varepsilon\cap R_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ *there exists a topological limit in the* η*-topology such that*

$$\eta\text{--Lm}\,(W_\varepsilon\cap R_\varepsilon) = \eta\text{--Lm}\, W_\varepsilon \cap \eta\text{--Lm}\, R_\varepsilon.$$

Proof. In accordance with Lemma 3 we need verify conditions (a)–(d) for the sets $W_\varepsilon = \mathrm{Gr}(\mathbf{A}_\varepsilon)$ and $R_\varepsilon = \widehat{Q}_\varepsilon \times H_0^1(\Omega)$, where $\widehat{Q}_\varepsilon$ are defined in Definition 2. Conditions (a)–(b) follow immediately from initial assumptions. Since the sequence of operators $\{\mathbf{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact with respect to G^*-convergence and the strong topology for $H^{-1}(\Omega)$ has a countable base, by the Kuratowski compactness theorem [11] there exist an index subset $H\in\mathbf{H}^\sharp$, a set $\emptyset\neq Q\subseteq H^{-1}(\Omega)$, and a coercive bounded operator $\mathbf{A}_*\in L(H_0^1(\Omega), H^{-1}(\Omega))$ such that

$$\tau\text{--Lm}\,\mathrm{Gr}(\mathbf{A}_\varepsilon) = \mathrm{gr}(\mathbf{A}_*),\ \varepsilon\in H;$$
$$\tau\text{--Lm}\left[\widehat{Q}_\varepsilon\times H_0^1(\Omega)\right] = \left[(s_{H^{-1}})-\mathrm{Lm}\,\widehat{Q}_\varepsilon\times H_0^1(\Omega)\right].$$

Therefore condition (c) of Lemma 3 holds. Finally, condition (d) follows immediately from supposition (iii). Hence, by Lemma 3 we have

$$\begin{aligned}\tau\text{--Lm}\left[\mathrm{Gr}(\mathbf{A}_\varepsilon)\Big|_{\widehat{Q}_\varepsilon\times H_0^1(\Omega)}\right] &= \tau\text{--Lm}\left(\mathrm{Gr}(\mathbf{A}_\varepsilon)\cap\left[\widehat{Q}_\varepsilon\times H_0^1(\Omega)\right]\right)\\ &= \tau\text{--Lm}\,[\mathrm{Gr}(\mathbf{A}_\varepsilon)]\cap\left[(s_{H^{-1}})-\mathrm{Lm}\,\widehat{Q}_\varepsilon\times H_0^1(\Omega)\right].\end{aligned}$$

This implies immediately (10).

Now, turning to the original homogenization problem, we introduce the following assumption (in addition to suppositions **(1)–(5)**):

(6) there exist linear mappings $J_\varepsilon : L^2(\Omega) \to L^2(\Omega)$ and a family of closed subsets $\left\{\widehat{U}_\varepsilon\right\}_{\varepsilon\in(0,\varepsilon_0]} \subseteq L^2(\Omega)$ such that

$$U_\varepsilon = \left\{u \in L^2(\Omega) \;\middle|\; u = J_\varepsilon v,\, v \in \widehat{U}_\varepsilon\right\} \text{ for every } \varepsilon \in (0, \varepsilon_0];$$

(7) there exists an invertible linear operator $J_0 : L^2(\Omega) \to L^2(\Omega)$ such that $J_\varepsilon \longrightarrow J_0$ in the weak operator topology, i.e. $\langle u, J_\varepsilon v\rangle_{L^2} \longrightarrow \langle u, J_0 v\rangle_{L^2}$ for every $u, v, \in L^2(\Omega)$, and the following inclusion holds $(w_{L^2})-\mathrm{Ls}\,\widehat{U}_\varepsilon \subseteq J_0^{-1}\left[(w_{L^2})-\mathrm{Lm}\,U_\varepsilon\right]$, where by $(w_{L^2})-\mathrm{Ls}\,\widehat{U}_\varepsilon$ is denoted the upper topological limit of the sequence $\left\{\widehat{U}_\varepsilon\right\}$;

(8) for every control sequence $\{u_\varepsilon \in U_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ weakly converging in $L^2(\Omega)$ there can be found a sequence of prototypes $\left\{v_\varepsilon \in \widehat{U}_\varepsilon\right\}_{\varepsilon\in(0,\varepsilon_0]}$ satisfying the conditions: $u_\varepsilon = J_\varepsilon v_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$ and $u_\varepsilon \longrightarrow u = J_0 v$ weakly in $L^2(\Omega)$, where $v \in L^2(\Omega)$ is the weak limit of $\left\{v_\varepsilon \in \widehat{U}_\varepsilon\right\}_{\varepsilon\in(0,\varepsilon_0]}$;

(9) for control object (3) hypotheses (A1)–(A4) hold true;

(10) for every $\varepsilon \in (0, \varepsilon_0]$ there exist a linear continuous operator $\widehat{\mathbf{B}}_\varepsilon$ from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and an element $\widehat{f}_\varepsilon \in H^{-1}(\Omega)$ such that:

$$R_\varepsilon^+ P_\varepsilon^* \left(\widehat{\mathbf{B}}_\varepsilon v + \widehat{f}_\varepsilon\right) = b_\varepsilon J_\varepsilon v + f_\varepsilon \text{ for every } v \in \widehat{U}_\varepsilon;$$

$$\widehat{f}_\varepsilon \to \widehat{f}_0 \text{ strongly in } H^{-1}(\Omega);$$

$$\widehat{\mathbf{B}}_\varepsilon \longrightarrow \widehat{\mathbf{B}}_0 \in L\left(L^2(\Omega); H^{-1}(\Omega)\right) \text{ in the uniform operator topology,}$$

$$\text{i.e. } \lim_{\varepsilon\to 0}\left\|\widehat{\mathbf{B}}_\varepsilon - \widehat{\mathbf{B}}_0\right\|_{L(L^2(\Omega);\, H^{-1}(\Omega))} = 0.$$

We begin with the following result.

LEMMA 4. *If assumptions* **(1)–(10)** *hold true, then*

$$\emptyset \neq (s_{H^{-1}})-\mathrm{Lm}\,\widehat{Q}_\varepsilon = \left\{g \in H^{-1}(\Omega) \;\middle|\; g = \widehat{\mathbf{B}}_0 J_0^{-1} u + \widehat{f}_0 \;\forall u \in (w_{L^2})-\mathrm{Lm}\,U_\varepsilon\right\}, \quad (11)$$

where $\widehat{f}_0$ *is a limit of* $\left\{\widehat{f}_\varepsilon\right\}$ *in the strong topology of* $H^{-1}(\Omega)$ *and* $\widehat{Q}_\varepsilon$ *are the convex closed subsets which are defined by the rule*

$$\widehat{Q}_\varepsilon = \left\{g \in H^{-1}(\Omega) \;\middle|\; g = \widehat{\mathbf{B}}_\varepsilon v + \widehat{f}_\varepsilon \;\forall v \in \widehat{U}_\varepsilon\right\}. \quad (12)$$

Proof. Let $g^* = \widehat{\mathbf{B}}_0 J_0^{-1} u^* + \widehat{f}_0$ be any element of the set

$$\left\{ g \in H^{-1}(\Omega) \;\middle|\; g = \widehat{\mathbf{B}}_0 J_0^{-1} u + \widehat{f}_0 \;\forall u \in (w_{L^2})\text{–}\mathrm{Lm}\, U_\varepsilon \right\}.$$

Then since $u^* \in (w_{L^2})\text{–}\mathrm{Lm}\, U_\varepsilon$, it follows that there exist an index set $H \in \mathbf{H}$, a sequence $\{u_\varepsilon^*\}_{\varepsilon \in H}$ converging to u^* in the weak topology of $L^2(\Omega)$, and a sequence of prototypes $\{v_\varepsilon^*\}_{\varepsilon \in H}$ weakly converging to v^* in $L^2(\Omega)$ such that

$$u_\varepsilon^* = J_\varepsilon v_\varepsilon^* \in U_\varepsilon, \quad v_\varepsilon^* \in \widehat{U}_\varepsilon \;\forall \varepsilon \in H \quad \text{and} \quad u^* = J_0 v^*.$$

Therefore, by property **(10)**, $\widehat{\mathbf{B}}_\varepsilon v_\varepsilon^* + \widehat{f}_\varepsilon \in \widehat{Q}_\varepsilon$ for every $\varepsilon \in H$. At the same time we have

$$\begin{aligned} \left\| \widehat{\mathbf{B}}_\varepsilon v_\varepsilon^* - \widehat{\mathbf{B}}_0 v^* \right\| &\le \left\| \left(\widehat{\mathbf{B}}_\varepsilon - \widehat{\mathbf{B}}_0 \right) v_\varepsilon^* \right\| + \left\| \widehat{\mathbf{B}}_0 \left(v_\varepsilon^* - v^* \right) \right\| \\ &\le \left\| \widehat{\mathbf{B}}_\varepsilon - \widehat{\mathbf{B}}_0 \right\| \cdot \|v_\varepsilon^*\| + \sup_{\|\phi\|_{H_0^1} = 1} \left\langle \widehat{\mathbf{B}}_0^* \phi, v_\varepsilon^* - v^* \right\rangle . \end{aligned}$$

Hence

$$\widehat{\mathbf{B}}_\varepsilon v_\varepsilon^* + \widehat{f}_\varepsilon \longrightarrow \widehat{\mathbf{B}}_0 v^* + \widehat{f}_0 = \widehat{\mathbf{B}}_0 J_0^{-1} u^* + \widehat{f}_0 \;\text{ strongly in }\; H^{-1}(\Omega).$$

On the other hand, if H be any index set of $\mathbf{H}^\sharp$ and $\left\{ g_\varepsilon \in \widehat{Q}_\varepsilon \right\}_{\varepsilon \in H}$ is a sequence converging to g in the strong topology of $H^{-1}(\Omega)$, then there is a sequence of control prototypes $\left\{ v_\varepsilon \in \widehat{U}_\varepsilon \right\}_{\varepsilon \in H}$ such that $g_\varepsilon = \widehat{\mathbf{B}}_\varepsilon v_\varepsilon + \widehat{f}_\varepsilon$ for every $\varepsilon \in H$. Since the sequence $\widehat{\mathbf{B}}_\varepsilon v_\varepsilon$ is bounded in $H^{-1}(\Omega)$ and the operators $\widehat{\mathbf{B}}_\varepsilon$ are compact with respect to the uniform operator topology, it follows the the sequence $\{v_\varepsilon\}_{\varepsilon \in H}$ is bounded as well. Hence we may assume that there is an element $v_0 \in \left(w_{L^2}\right) \text{–}\mathrm{Ls}\, \widehat{U}_\varepsilon$ such that $v_\varepsilon \longrightarrow v_0$ weakly in $L^2(\Omega)$. Consequently,

$$\begin{aligned} g_\varepsilon = \widehat{\mathbf{B}}_\varepsilon v_\varepsilon + \widehat{f}_\varepsilon \in \widehat{Q}_\varepsilon \qquad & \text{for every } \varepsilon \in H; \\ g_\varepsilon \longrightarrow \widehat{\mathbf{B}}_0 v_0 + \widehat{f}_0 = g_0 \quad & \text{strongly in } H^{-1}(\Omega). \end{aligned}$$

But by property **(7)** there can be found an element u_0 in $\left(w_{L^2}\right) \text{–}\mathrm{Lm}\, U_\varepsilon$ satisfying $v_0 = J_0^{-1} u_0$. Therefore $g_0 = \widehat{\mathbf{B}}_0 J_0^{-1} u_0 + \widehat{f}_0$. Thus, by Lemma 1, we obtain the required.

Now we are in a position to state the main result of our paper.

THEOREM 3. *Suppose that conditions* **(1)–(10)** *hold true and there is an index set* $H \in \mathbf{H}$ *and some* μ*-converging sequence of admissible pairs* $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon \in H}$ *for original control problem* (1)–(2). *Then for the sequence of sets of admissible pairs* $\{\Xi_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$ *there exists a topological limit in the* μ*-topology and one has the following representation*

$$\mu\text{–}\mathrm{Lm}\, \Xi_\varepsilon = \mathbb{X}, \tag{13}$$

where

$$\mathbb{X} = \left\{ (u,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{c} \mathbf{A}_* y = \widehat{\mathbf{B}}_0 J_0^{-1} u + \widehat{f}_0 \,, \\ u \in \left(w_{L^2}\right) - \mathrm{Lm}\, U_\varepsilon . \end{array} \right\},$$

where $\mathbf{A}_* \in L\left(H_0^1(\Omega); H^{-1}(\Omega)\right)$ *is the* G^* *-limit of the sequence of operators* $\{\mathbf{A}_\varepsilon\}$ *in the sense of Definition 4.*

Proof. First of all we note that by initial assumptions there is some sequence of admissible pair $\{(u_\varepsilon, y_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon \in H}$ such that $(u_\varepsilon, y_\varepsilon) \xrightarrow{\mu} (u^0, y^0)$. However, by property **(8)** there can be found a sequence of control prototypes $\left\{v_\varepsilon \in \widehat{U}_\varepsilon\right\}_{\varepsilon \in (0,\varepsilon_0]}$ satisfying the conditions: $u_\varepsilon = J_\varepsilon v_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_0]$ and $u_\varepsilon \longrightarrow u^0 = J_0 v^0$ weakly in $L^2(\Omega)$, where $v^0 \in L^2(\Omega)$ is the weak limit of $\left\{v_\varepsilon \in \widehat{U}_\varepsilon\right\}_{\varepsilon \in (0,\varepsilon_0]}$. Therefore in view of condition **(10)** instead of the original sequence of admissible pairs we may consider the sequence of their prototypes $\left\{(v_\varepsilon, y_\varepsilon) \in \widehat{\Xi}_\varepsilon\right\}_{\varepsilon \in H}$, where the sets $\widehat{\Xi}_\varepsilon$ are defined by the rule

$$\widehat{\Xi}_\varepsilon = \left\{ (v,y) \in L^2(\Omega) \times H_0^1(\Omega) \;\middle|\; \begin{array}{c} \mathbf{A}_\varepsilon y = R_\varepsilon^+ P_\varepsilon^* \left(\widehat{\mathbf{B}}_\varepsilon v + \widehat{f}_\varepsilon \right), \\ v \in \widehat{U}_\varepsilon . \end{array} \right\}$$

Consequently, by Lemma 4 and condition **(8)**, we have

$$\widehat{Q}_\varepsilon \ni \widehat{\mathbf{B}}_\varepsilon v_\varepsilon + \widehat{f}_\varepsilon \longrightarrow \widehat{\mathbf{B}}_0 J_0^{-1} u^0 + \widehat{f}_0 \in \left(s_{H^{-1}}\right) - \mathrm{Lm}\, \widehat{Q}_\varepsilon \quad \text{strongly in } H^{-1}(\Omega),$$

i.e. all suppositions on Theorem 2 hold true. Therefore for the topological limit of prototype graph restrictions $\left[\mathrm{Gr}\,(\mathbf{A}_\varepsilon)\Big|_{\widehat{Q}_\varepsilon \times H_0^1(\Omega)}\right]$ representation (10) holds.

Let $(\widehat{u}^*, \widehat{y}^*)$ be any pair of $\mathbb{X}$. Then, by Lemma 4, we have

$$\widehat{g}^* = \widehat{\mathbf{B}}_0 J_0^{-1} \widehat{u}^* + \widehat{f}_0 \in \left(s_{H^{-1}}\right) - \mathrm{Lm}\, \widehat{Q}_\varepsilon,$$

where the sets $\widehat{Q}_\varepsilon$ are defined in (12). Using Theorem 2 we deduce that

$$(\widehat{g}^*, \widehat{y}^*) \in \mathrm{gr}\,(\mathbf{A}_*) \cap \left[\left(s_{H^{-1}}\right) - \mathrm{Lm}\, \widehat{Q}_\varepsilon \times H_0^1(\Omega) \right].$$

Here $\mathbf{A}_*$ is the G^*-limit of the operators sequence $\{\mathbf{A}_\varepsilon\}_{\varepsilon \in (0,\varepsilon_0]}$. Then in accordance with Theorem 2 we obtain

$$\begin{aligned} (\widehat{g}^*, \widehat{y}^*) \in \; & \tau\text{--}\mathrm{Lm}\, \mathrm{Gr}\,(\mathbf{A}_\varepsilon) \cap \left[\left(s_{H^{-1}}\right) - \mathrm{Lm}\, \widehat{Q}_\varepsilon \times H_0^1(\Omega) \right] \\ & = \tau\text{--}\mathrm{Lm} \left[\mathrm{Gr}\,(\mathbf{A}_\varepsilon) \Big|_{\widehat{Q}_\varepsilon \times H_0^1(\Omega)} \right]. \end{aligned}$$

Therefore, by properties of topological limits (see Lemma 1), there exist an index set $H \in \mathbf{H}$, and sequences $\{\widehat{y}_\varepsilon\}_{\varepsilon \in H}$, $\{\widehat{u}_\varepsilon\}_{\varepsilon \in H}$, and $\{\widehat{v}_\varepsilon\}_{\varepsilon \in H}$ such that

$$\begin{array}{rll} \widehat{y}_\varepsilon \longrightarrow \widehat{y}^* & \text{weakly in} & H_0^1(\Omega), \\ \widehat{U}_\varepsilon \ni \widehat{v}_\varepsilon \longrightarrow \widehat{v}^* & \text{weakly in} & L^2(\Omega), \\ U_\varepsilon \ni J_\varepsilon \widehat{v}_\varepsilon = \widehat{u}_\varepsilon \longrightarrow \widehat{u}^* = J_0 \widehat{v}^* & \text{weakly in} & L^2(\Omega), \\ \widehat{Q}_\varepsilon \ni \widehat{g}_\varepsilon = \widehat{\mathbf{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon \longrightarrow \widehat{\mathbf{B}}_0 J_0^{-1} \widehat{u}^* + \widehat{f}_0 = \widehat{g}^* & \text{strongly in} & H^{-1}(\Omega), \\ \mathbf{A}_\varepsilon \widehat{y}_\varepsilon = R_\varepsilon^+ P_\varepsilon^* \widehat{g}_\varepsilon = b_\varepsilon \widehat{u}_\varepsilon + f_\varepsilon & \text{for every} & \varepsilon \in H. \end{array}$$

Thus for the pair $(\widehat{u}^*,\widehat{y}^*)$ we have found the insex set $H \in \mathbf{H}$ and constructed the sequence $\{(\widehat{u}_\varepsilon,\widehat{y}_\varepsilon)\}_{\varepsilon\in H}$ such that

$$(\widehat{u}_\varepsilon,\widehat{y}_\varepsilon) \xrightarrow{\mu} (\widehat{u}^*,\widehat{y}^*) \text{ and } (\widehat{u}_\varepsilon,\widehat{y}_\varepsilon) \in \Xi_\varepsilon \text{ for every } \varepsilon \in H,$$

i.e. condition (i) of Lemma 1 holds.

Now we consider any index set H of $\mathbf{H}^\sharp$. Let $\{(\widehat{u}_\varepsilon,\widehat{y}_\varepsilon)\}_{\varepsilon\in H}$ be a sequence μ-converging to some pair (u,y) such that $(\widehat{u}_\varepsilon,\widehat{y}_\varepsilon) \in \Xi_\varepsilon$ for every $\varepsilon \in H$. We have to show that $(u,y) \in \mathbb{X}$. Indeed, in this case there can be found a sequence of prototypes $\{\widehat{v}_\varepsilon\}_{\varepsilon\in H}$ weakly converging to v in $L^2(\Omega)$ such that

$$\widehat{u}_\varepsilon = J_\varepsilon \widehat{v}_\varepsilon \in U_\varepsilon, \quad \widehat{v}_\varepsilon \in \widehat{U}_\varepsilon \ \forall \varepsilon \in H \quad \text{and} \quad u = J_0 v.$$

Consequently,

$$\begin{aligned} \widehat{g}_\varepsilon = \widehat{\mathbf{B}}_\varepsilon \widehat{v}_\varepsilon + \widehat{f}_\varepsilon \longrightarrow \widehat{\mathbf{B}}_0 J_0^{-1} u + \widehat{f}_0 = \widehat{g}_0 \quad & \text{strongly in } H^{-1}(\Omega), \\ \widehat{y}_\varepsilon = \mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* \widehat{g}_\varepsilon \longrightarrow y \quad & \text{weakly in } H_0^1(\Omega), \end{aligned}$$

and by virtue of Theorem 2 we have

$$(\widehat{g}_0, y) \in \mathrm{gr}\,(\mathbf{A}_*) \Big|_{(s_{H^{-1}})-\mathrm{Lm}\,\widehat{Q}_\varepsilon \times H_0^1(\Omega)}.$$

Therefore $y = \mathbf{A}_*^{-1}\widehat{g}_0 = \mathbf{A}_*^{-1}\left(\widehat{\mathbf{B}}_0 J_0^{-1} u + \widehat{f}_0\right)$, i.e. we have the following inclusion

$$(u,y) \in \mathbb{X}.$$

Thus, using Lemma 1, we deduce that the set $\mathbb{X}$ is the topological limit of the sequence of sets of admissible pairs $\{\Xi_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$. The proof is complete.

We have proved that under initial assumptions **(1)–(10)** there exists the homogenized control object for (1)–(2) and this one can be presented in the following form:

$$\mathbf{A}_* y = \widehat{\mathbf{B}}_0 J_0^{-1} u + \widehat{f}_0 \quad \text{in} \quad D'(\Omega),$$

$$u \in (w_{L^2}) - \mathrm{Lm}\, U_\varepsilon.$$

In conclusion we give the example which shows that in the genaral case the G^*-limit $\mathbf{A}_*$ of the operators $\{\mathbf{A}_\varepsilon\}$ may not coincide with G-limit $\mathbf{A}_0$ of such a sequence.

Let Ω be an open bounded domain of R^n, and let $\{\Omega_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a sequence of open domains of R^n which are contained in Ω. Let $\{\mathbf{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a sequence of linear uniformly coercive and uniformly bounded operators from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. For every $\varepsilon \in (0,\varepsilon_0]$ we put

(i) L^ε be the closure in $H^{-1}(\Omega)$ of the set of all functions $f \in C^\infty(\Omega)$ with $\mathrm{supp}\, f$ contained in Ω_ε;

(ii) $Y_\varepsilon = H_0^1(\Omega_\varepsilon)$;

(iii) $P_\varepsilon : H_0^1(\Omega_\varepsilon) \to H_0^1(\Omega)$ be the extention operator defined for every $y \in H_0^1(\Omega_\varepsilon)$ by $(P_\varepsilon y)|_{\Omega_\varepsilon} = y$, $(P_\varepsilon y)|_{\Omega\setminus\Omega_\varepsilon} = 0$. Since P_ε is linear continuous operator, the conjugate operator $P_\varepsilon^* : H^{-1}(\Omega) \to H^{-1}(\Omega_\varepsilon)$ is defined;

(iv) $R_\varepsilon^+ : H^{-1}(\Omega_\varepsilon) \to (L^\varepsilon \subset H^{-1}(\Omega))$ be the extention operator defined for every $f \in H^{-1}(\Omega_\varepsilon)$ by $(R_\varepsilon^+ f)|_{\Omega_\varepsilon} = f$, $(R_\varepsilon^+ y)|_{\Omega\setminus\Omega_\varepsilon} = 0$.

Assume that Kovalevsky's hypothesis holds: each of operators $\{\mathbf{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ has the following representation

$$\mathbf{A}_\varepsilon^{-1} = P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^*,$$

where $\Lambda_\varepsilon \in L(Y_\varepsilon; Y_\varepsilon^*)$ are some invertible operators and if $y \in C_0^\infty(\Omega)$ then there exist a constant $\nu > 0$ and a sequence $\{y_\varepsilon \in K_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ such that $y_\varepsilon \to y$ weakly in $H_0^1(\Omega)$ and such that, for every closed cube $S \subset \Omega$,

$$\limsup_{\varepsilon\to 0} \int_S |\nabla y_\varepsilon|^2\, dx \le \nu \int_S \left(|\nabla y|^2 + y^2\right) dx,$$

where by K_ε we denote the closure in $H_0^1(\Omega)$ of the set of all functions $y \in C^\infty(\Omega)$ with $\operatorname{supp} y$ contained in Ω_ε.

Then $\mathbf{A}_\varepsilon \xrightarrow{G^*} \mathbf{A}_*$ if and only if

$$\mathbf{A}_\varepsilon^{-1} R_\varepsilon^+ P_\varepsilon^* f = \left[P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^*\right] R_\varepsilon^+ P_\varepsilon^* f \equiv P_\varepsilon \Lambda_\varepsilon^{-1} P_\varepsilon^* f \longrightarrow \mathbf{A}_*^{-1} f \quad \text{weakly in } H_0^1(\Omega)$$

for every $f \in H^{-1}(\Omega)$.

Therefore in view of Kovalevsky's theorem (see [11]) we deduce that for the G^*-limit operator $\mathbf{A}_*$ the following representation holds:

$$\mathbf{A}_* = \mathbf{A}_0 + F_\mu,$$

where $\mathbf{A}_0$ is the G-limit of $\{\mathbf{A}_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ in the usual sense, and the operator $F_\mu : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by

$$\langle F_\mu y, z\rangle = \int_\Omega \mu(x) y z\, dx.$$

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